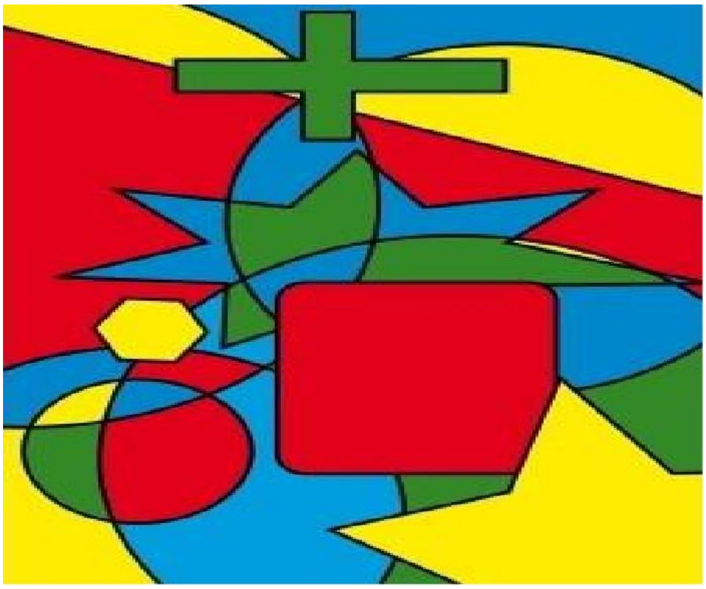
BCS405A

Discrete Mathematical Structures

(For the 4th Semester Computer Science and Engineering Stream)



Module 1

Mathematical Logic

Content

Module-1

Mathematical Logic

☻ **Syllabus:**

Fundamentals of Logic: Basic Connectives and Truth Tables, Logic Equivalence – The Laws of Logic, Logical Implication – Rules of Inference. The Use of Quantifiers, Quantifiers, Definitions and the Proofs of Theorems.

☻ **Basic Connectives and Truth table:**

Proposition:

A proposition is a declarative sentence that is either true or false, but not both.

Example:

1. 2 is a prime number. (true)
2. All sides are equal in scalene triangle. (false)
3. 2+3=4. (false)
4. What is the time now?
5. Read this carefully.

From the above examples we note that 1, 2, 3 are proposition, whereas 4 and 5 are not the propositions.

Logical Connectives and Truth table:

New propositions are obtained by starting with given propositions with the aid of words or phrases like 'not', 'and', 'if … then, and 'if and only if'. Such words or phrases are called **Logical connectives**.

1. Negation:

A proposition is obtained by inserting the word 'not' at an appropriate place in the given proposition is called the negation of the given proposition.

The negation of a Proposition p is denoted by $\neg p$ (read 'not p'). For any Proposition p, if p is true, then $\neg p$ is false, and if p is false, then $\neg p$ is true. i.e., If the truth value of a proposition p is 1 then the truth value of $\neg p$ is 0 and If the truth value of a proposition p is 0 then the truth value of $\neg p$ is 1.

Example:

p: 4 is an even number.

$\neg p$: 4 is not an even number.

Truth table for Negation

p	¬ p
0	1
1	0

2. **Conjunction:**

A compound proposition obtained by combining two given propositions by inserting the word ‘and’ in between them is called the conjunction of the given proposition.

The conjunction of two propositions p and q is denoted by p ∧ q (read ‘p and q’). The conjunction p ∧ q is true only when p is true and q is true, in all other cases it is false. i.e., the truth value of the conjunction p ∧ q is 1 only when the truth value of p is 1 and truth value of q is 1, in all other cases the truth value of p ∧ q is 0.

Example:

p: $\sqrt{2}$ is an irrational number.

q: 9 is a prime number.

p ∧ q: $\sqrt{2}$ is an irrational number and 9 is a prime number.

Truth table for conjunction

p	q	p ∧ q
0	0	0
0	1	0
1	0	0
1	1	1

3. **Disjunction:**

A compound proposition obtained by combining two given propositions by inserting the word ‘or’ in between them is called the disjunction of the given propositions.

The disjunction of two propositions p and q is denoted by p ∨ q (read ‘p or q’). The disjunction p ∨ q is false only when p is false and q is false, in all other cases it is true. i.e., the truth value of the disjunction p ∨ q is 0 only when the truth value of p is 0 and truth value of q is 0, in all other cases the truth value of p ∨ q is 1.

Example:

p: All triangles are equilateral.

q: 2+5=7.

$p \vee q$: All triangles are equilateral or 2+5=7.

Truth table for Disjunction

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

4. **Exclusive Disjunction:**

We require that the compound proposition "p or q" to be true only when either p is true or q is true but not both. The exclusive or is denoted by the $\veebar$.

The compound proposition $p \veebar q$ (read as either p or q but not both) is called as exclusive disjunction of the propositions p and q. i.e., $p \veebar q = (p \wedge \neg q) \vee (q \wedge \neg p)$

Example:

p:9 is a prime number

q: all triangles are isosceles.

$p \veebar q$: Either 9 is prime number or all triangles are isosceles, but not both

Truth table for Exclusive Disjunction

p	q	$p \veebar q$
0	0	0
0	1	1
1	0	1
1	1	0

5. **Conditional:**

A compound proposition obtained by combining two given propositions by using the words 'if' and 'then' at appropriate places is called a conditional.

The Conditional "If p, then q" is denoted by $p \rightarrow q$ and the Conditional "If q, then p" is denoted by $q \rightarrow p$. The Conditional $p \rightarrow q$ is false only when p is true and q is false, in all other cases it is true. i.e., the truth value of the conditional $p \rightarrow q$ is 0 only when the truth value of p is 1 and the truth value of q is 0, in all other cases the truth value of $p \rightarrow q$ is 1.

Example:

p: 3 is a prime number.

q: 9 is a multiple of 6

Truth table for Conditional

6.

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

<u>Biconditional:</u>

Let p and q be two sample propositions then the conjunction of the conditionals $p \rightarrow q$ and $q \rightarrow p$ is called the biconditional of p and q. It is denoted by $p \leftrightarrow q$ and it is same as $(p \rightarrow q) \wedge (q \rightarrow p)$ is read as "If p then q and if q then p".

Truth table for Biconditional

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Problems:

1. Construct the truth tables for the following propositions.

(i). $p \wedge (\neg q)$ **(ii).** $(\neg p) \vee q$ **(iii).** $p \rightarrow (\neg q)$ **(iv).** $(\neg p) \veebar (\neg q)$

Solution:

The desired truth tables are obtained by considering all possible combinations of the truth values of p and q. the combined form of required truth table is given below

p	q	$\neg p$	$\neg q$	$p \wedge (\neg q)$	$(\neg p) \vee q$	$p \rightarrow (\neg q)$	$(\neg p) \veebar (\neg q)$
0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	1	1
1	1	0	0	0	1	0	0

2. Let p, q and r be propositions having truth values 0, 0 and 1 respectively. Find the truth values if the following compound propositions:

(i). $(p \vee q) \vee r$ **(ii).** $(p \wedge q) \wedge r$ **(iii).** $(p \wedge q) \rightarrow r$

(iv). $p \rightarrow (q \wedge r)$ **(v).** $p \wedge (q \rightarrow r)$ **(vi).** $p \rightarrow (q \rightarrow \neg r)$

Solution:

(i) Since both p and q are false then $(p \vee q)$ is also false. Since r true it follows that $(p \vee q) \vee r$ is true. Thus, the truth value of $(p \vee q) \vee r$ is 1.

(ii) Since both p and q are false, $(p \wedge q)$ is false. Since $(p \wedge q)$ is false and r is true $(p \wedge q) \wedge r$ is false. Thus, the truth value of $(p \wedge q) \wedge r$ is 0.

(iii) Since $(p \wedge q)$ is false and r is true, $(p \wedge q) \rightarrow r$ is true. Thus, the truth value of $(p \wedge q) \rightarrow r$ is 1.

(iv) Since q is false and r is true, $(q \wedge r)$ is false. Also, p is false, therefore $p \rightarrow (q \wedge r)$ is true. Thus, the truth value of $p \rightarrow (q \wedge r)$ is 1.

(v) Since r is true and q is false $(q \rightarrow r)$ is true. Also, p is false. Therefore, $p \wedge (q \rightarrow r)$ is false. Thus, the truth value of $p \wedge (q \rightarrow r)$ is 0

(vi) Since r is true, $\neg r$ is false. Since q is false, $q \rightarrow (\neg r)$ is true. Also, p is false. Therefore, $p \rightarrow (q \rightarrow \neg r)$ is true. Thus, the truth value of $p \rightarrow (q \rightarrow \neg r)$ is 1.

3. Indicate how many rows are needed in the truth table for the compound proposition $(p \vee (\neg q)) \leftrightarrow ((\neg r) \wedge s) \rightarrow t$. Find the truth value of the proposition if p and r, are true and q, s, t, are false.

Solution:

The given compound proposition contains five primitives p, q, r, s, t. Therefore, the number of possible combinations of the truth values of these components which we have to consider is $2^5=32$. Hence 32 rows are needed in the truth table for the given compound proposition.

Next, suppose that p and r, are true and q, s, t are false, then $\neg q$ is true and $\neg r$ is false. Since p is true and $\neg q$ is true, $(p \vee (\neg q))$ is true on the other hand, since $\neg r$ is false and s is false, $\neg r \wedge s$ is false. Also, t is false. Hence $((\neg r) \wedge s) \rightarrow t$ is true.

Since $(p \vee (\neg q))$ is true and $((\neg r) \wedge s) \rightarrow t$ is true, it follows that the truth values of the given propositions $(p \vee (\neg q)) \leftrightarrow ((\neg r \wedge s) \rightarrow t$ is 1.

4. Let p: A circle is a conic, q: $\sqrt{5}$ is a real number, r: Exponential series is convergent. Express the following compound Proposition in words:
(i). $p \wedge (\neg q)$ **(ii).** $(\neg p) \wedge q$ **(iii)** $q \rightarrow (\neg p)$
(iv). $p \veebar (\neg q)$ **(v).** $p \rightarrow (q \veebar r)$ **(vi).** $\neg p \leftrightarrow q$

Solution:

(i) A circle is a conic and $\sqrt{5}$ is not a real number.

(ii) A circle is not a conic and $\sqrt{5}$ is a real number.

(iii) If $\sqrt{5}$ is a real number, then a circle is not a conic.

(iv) Either a circle is a conic or $\sqrt{5}$ is not a real number (but not both).

(v) If a circle is a conic then either $\sqrt{5}$ is a real number or the exponential series is convergent (but not both).

(vi) If a circle is not a conic then $\sqrt{5}$ is a real number and if $\sqrt{5}$ is a real number then a circle is not a conic.

5. Construct the truth table for the following compound propositions:
(i). $(p \wedge q) \rightarrow \neg r$ **(ii).** $q \wedge ((\neg r) \rightarrow p)$

Solution:

The required truth table are shown below in a combined form

p	q	r	$\neg r$	$p \wedge q$	$(p \wedge q) \rightarrow \neg r$	$(\neg r) \rightarrow p$	$q \wedge ((\neg r) \rightarrow p)$
0	0	0	1	0	1	0	0
0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	1	1
1	0	0	1	0	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	0	1	0	1	1

☻ Tautology and Contradiction:

A compound proposition which is true for all possible truth values of its components is called a **tautology**.
A compound proposition which is false for all possible truth values of its components is called **Contradiction or an absurdity.**
A compound proposition that can be true or false is called a **contingency.** In other words, a contingency is a compound proposition which is neither a tautology nor a contradiction.

Problems:

1. Show that for any proposition p and q, the compound proposition $p \rightarrow (p \vee q)$ is a tautology and the compound proposition $p \wedge (\neg p \wedge q)$ is called contradiction.

Solution:

Let us first prepare the truth tables for $p \rightarrow (p \vee q)$ and $p \wedge (\neg p \wedge q)$. these truth tables are shown below in the combined form.

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\neg p$	$(\neg p \wedge q)$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

From the above table we note that, for all possible values of p and q the compound proposition $p \rightarrow (p \vee q)$ is true and the compound proposition $p \wedge (\neg p \wedge q)$ is false.

Therefore $p \wedge (\neg p \wedge q)$ is **contradiction** and $p \rightarrow (p \vee q)$ is **tautology**.

2. Prove that, for any proposition p, q, r the compound proposition
$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology

Solution:
The following truth table gives the required result.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

3. Prove that for any proposition p, q, r the compound proposition

$(p \vee q) \vee (p \to r) \wedge (q \to r)$ is tautology.

Solution:

The following truth table gives the required result.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$p \vee q$	$(p \vee q) \vee (p \to r) \wedge (q \to r)$
0	0	0	1	1	1	0	1
0	0	1	1	1	1	0	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	1	1
1	0	1	1	1	1	1	1
1	1	0	0	0	0	1	1
1	1	1	1	1	1	1	1

4. Prove that for any proposition p, q, r the compound proposition

$(p \to q) \vee (p \to r) \leftrightarrow (p \to (q \vee r))$ is tautology.

Solution:
The following truth table gives the required result.

p	q	r	p → q	p → r	(p → q) ∨ (p → r)	q ∨ r	p → (q ∨ r)	(p →q) ∨ (p → r) ↔ (p → (q ∨ r))
0	0	0	1	1	1	0	1	1
0	0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	1
1	0	1	0	1	1	1	1	1
1	1	0	1	0	1	1	1	1
1	1	1	1	1	1	1	1	1

5. Prove that for any proposition p, q, r the compound proposition
[(p → q) ∧ (p → r)] → (p → r)) is tautology.

Solution:
The following truth table gives the required result.

p	q	r	p → q	q → r	(p → q) ∧ (p → r)	p → r	(p →q) ∧ (p → r) → (p → r)
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

6. Prove that for any proposition p, q, r the compound proposition
[(p ∨ q) ∧ {(p → r) ∧ (q → r)}] → r is tautology.

Solution:
The following truth table proves the gives result.

p	q	r	p → r	q → r	(p → r) ∧ (q → r)	p ∨ q	(p ∨ q) ∧ {(p → r) ∧ (q → r)}	[(p ∨ q) ∧ {(p → r) ∧ (q → r)}] → r
0	0	0	1	1	1	0	0	1
0	0	1	1	1	1	0	0	1
0	1	0	1	0	0	1	0	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	0	1
1	0	1	1	1	1	1	1	1
1	1	0	0	0	0	1	0	1
1	1	1	1	1	1	1	1	1

7. Verify the Compound Proposition (p ∨ q) → r ↔ (¬ r → ¬ (p ∨ q)) is tautology or not.

p	q	r	¬ r	p ∨ q	(p ∨ q) → r	¬ (p ∨ q)	¬ r → ¬ (p ∨ q)	(p ∨ q) → r ↔ (¬ r → ¬ (p ∨ q))
0	0	0	1	0	1	1	1	1
0	0	1	0	0	1	1	1	1
0	1	0	1	1	0	0	0	1
0	1	1	0	1	1	0	1	1
1	0	0	1	1	0	0	0	1
1	0	1	0	1	1	0	1	1
1	1	0	1	1	0	0	0	1
1	1	1	0	1	1	0	1	1

Hence the compound Proposition (p ∨ q) → r ↔ (¬ r → ¬ (p ∨ q)) is tautology

8. Prove that for any proposition p, q, r the compound proposition

$\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$ is tautology.

Solution:

The following truth table gives the required result.

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r)$	$\{(p \to q) \to (p \to r)$	$\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	1	1	0	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

☻ **Logic equivalence:**

Two statement s_1, s_2 are said to be logically equivalent, and we write $s_1 \leftrightarrow s_2$, when the statement s_1 is true (respectively false) if and only if the statement s_2 is true (respectively false). Or the biconditional $s_1 \leftrightarrow s_2$ is a tautology

Problems:

1. For any two propositions p, q Prove that $(p \to q) \Leftrightarrow (\neg p) \vee q$

Solution: The following truth table gives the required result.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

From the column 4 and 5 of the above truth table, we find that $\neg p \vee q$ and $p \to q$ has the same truth values of p and q. Therefore $(p \to q) \Leftrightarrow (\neg p) \vee q$.

2. For any two propositions p, q Prove that $(p \to \neg q) \Leftrightarrow (q \to \neg p)$

Solution: The following truth table gives the required result.

p	q	$\neg p$	$\neg q$	$p \to \neg q$	$q \to \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

From the column 5 and 6 of the above truth table, we find that $p \to \neg q$ and $q \to \neg p$ has the same truth values of p and q. Therefore $(p \to \neg q) \Leftrightarrow (q \to \neg p)$.

3. For any two propositions p, q Prove that $(p \veebar q) \Leftrightarrow (p \vee q) \wedge \neg (p \wedge q)$.

Solution: The following truth table gives the required result.

p	q	$(p \vee q)$	$(p \veebar q)$	$(p \wedge q)$	$\neg (p \wedge q)$	$(p \vee q) \wedge \neg (p \wedge q)$
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	1	0	1	0	0

From the column 4 and 7 of the above truth table, we find that $(p \veebar q)$ and $(p \vee q) \wedge \neg (p \wedge q)$ has the same truth values of p and q. Therefore $(p \veebar q) \Leftrightarrow (p \vee q) \wedge \neg (p \wedge q)$.

4. For any propositions p, q, r. Prove that $[(p \rightarrow (q \rightarrow r)] \Leftrightarrow [(p \wedge \neg r) \rightarrow \neg q)]$

Solution: The following truth table gives the required result.

p	q	r	$\neg q$	$\neg r$	$q \rightarrow r$	$p \wedge \neg r$	$p \rightarrow (q \rightarrow r)$	$(p \wedge \neg r) \rightarrow \neg q)$
0	0	0	1	1	1	0	1	1
0	0	1	1	0	1	0	1	1
0	1	0	0	1	0	0	1	1
0	1	1	0	0	1	0	1	1
1	0	0	1	1	1	1	1	1
1	0	1	1	0	1	0	1	1
1	1	0	0	1	0	1	0	0
1	1	1	0	0	1	0	1	1

From the column 8 and 9 of the above truth table, we find that $[p \rightarrow (q \rightarrow r)]$ and $[(p \wedge \neg r) \rightarrow \neg q)]$ has the same truth values of p and q. Therefore $[(p \rightarrow (q \rightarrow r)] \Leftrightarrow [(p \wedge \neg r) \rightarrow \neg q)]$.

5. Show that the compound propositions $p \wedge ((\neg q) \vee r)$ and $p \vee (q \wedge (\neg r))$ are not logically equivalent.

Solution: The following truth table gives the required result

p	q	r	$\neg q$	$\neg r$	$\neg q \vee r$	$q \wedge \neg r$	$p \wedge ((\neg q) \vee r)$	$p \vee (q \wedge (\neg r))$
0	0	0	1	1	1	0	0	0
0	0	1	1	0	1	0	0	0
0	1	0	0	1	0	1	0	1
0	1	1	0	0	1	0	0	0
1	0	0	1	1	1	0	1	1
1	0	1	1	0	1	0	1	1
1	1	0	0	1	0	1	0	1
1	1	1	0	0	1	0	1	1

From the last two rows we note that $p \wedge ((\neg q) \vee r)$ and $p \vee (q \wedge (\neg r))$ do not have the same values in all possible situations. Therefore, they are not logically equivalent.

<u>The Laws of Logic:</u>

For any primitive statements p, q, r any tautology T_o and any contradiction F_o

Sl. No	Name of laws	Laws of logic
1	Laws of double negation	$\neg \neg p \Leftrightarrow p$
2	De Morgan's laws	$\neg (p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ $\neg (p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$
3	Commutative laws	$(p \vee q) \Leftrightarrow (q \vee p)$ $(p \wedge q) \Leftrightarrow (q \wedge p)$
4	Associative laws	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
5	Distributive laws	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
6	Idempotent laws	$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$
7	Identity laws	$p \vee F_o \Leftrightarrow p$ $p \wedge T_o \Leftrightarrow p$
8	Inverse laws	$p \vee \neg p \Leftrightarrow T_o$ $p \wedge \neg p \Leftrightarrow F_o$
9	Domination laws	$p \vee T_o \Leftrightarrow T_o$ $p \wedge F_o \Leftrightarrow F_o$
10	Absorption laws	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$

Problems:

1. Prove distributive law $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Solution:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	0	1	1	1	1

From columns 5 and 8 of the above table, we find that $\{p \vee (q \wedge r)\}$ and $\{(p \vee q) \wedge (p \vee r)\}$ has same truth values in all possible situations. Therefore, $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

Similarly, we can prove $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$.

2. Prove De Morgan's law $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Solution:

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg (p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

From columns 5 and 8 of the above table, we find that $\neg (p \vee q)$ and $\neg p \vee \neg q$ has same truth values in all possible situations. Therefore, $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$.

Similarly, we can prove $\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Law for the negation of a conditional:

Given a conditional $p \rightarrow q$, its negation is obtained by using the following law.

$\neg (p \rightarrow q) \Leftrightarrow [p \wedge (\neg q)]$

Proof:

The following table gives the truth values of $\neg (p \rightarrow q)$ and $p \wedge (\neg q)$ for all possible truth values of p and q.

p	q	$p \rightarrow q$	$\neg (p \rightarrow q)$	$\neg q$	$p \wedge (\neg q)$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

We note that $\neg (p \rightarrow q)$ and $p \wedge (\neg q)$ have same truth values in all possible situations. Hence, $\neg (p \rightarrow q) \Leftrightarrow [p \wedge (\neg q)]$.

Problems:

1. Simplify the following compounds propositions using the laws of logic.

(i) $p \vee q \wedge [\neg \{(\neg p) \wedge q\}]$ **(ii)** $p \vee q \wedge [\neg \{(\neg p) \vee q\}]$

(iii) $\neg [\neg \{(p \vee q) \wedge r\} \vee (\neg q)]$

Solution:

(i) $p \vee q \wedge [\neg \{(\neg p) \wedge q\}]$

$= p \vee q \wedge \{(\neg \neg p) \vee (\neg q)\}$ By De Morgan's law

$= p \vee q \wedge \{p \vee (\neg q)\}$ By Law of double negation

$= p \vee \{q \wedge (\neg q)\}$ By Distributive law

$= p \vee F_o$ By Inverse law

$= p$ By Identity law

(ii) $p \vee q \wedge [\neg \{(\neg p) \vee q\}]$

$= (p \vee q) \wedge \{p \wedge (\neg q)\}$

$= \{(p \vee q) \wedge p)\} \wedge (\neg q)$ Using Associative law

$= \{p \wedge (p \vee q)\} \wedge (\neg q)$ — Using Commutative law

$= p \wedge (\neg q)$ — Using Absorption law

(iii) $\neg [\neg \{(p \vee q) \wedge r\} \vee (\neg q)]$

$= \neg [\neg \{((p \vee q) \wedge r) \wedge q\}]$ — Using De Morgan's law

$= ((p \vee q) \wedge r) \wedge q$ — Law of Double negation

$= (p \vee q) \wedge (q \wedge r)$ — Using Associative and Commutative law

$= \{(p \vee q) \wedge q\} \wedge r$ — Using Associative law

$= q \wedge r$ — Using Associative law

2. Prove the following logically without using truth table.

(i). $[p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow p \vee q \vee r$ **(ii).** $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

(iii). $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

Solution:

(i) $[p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow p \vee q \vee r$

We have, $\neg p \wedge \neg q \wedge r \Leftrightarrow \neg (p \vee q) \wedge r$ — By De Morgan's law

Therefore, $[p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q)$ v $(\neg (p \vee q) \wedge r)$

$\Leftrightarrow [(p \vee q) \vee \neg (p \vee q)] \wedge (p \vee q \vee r)$ — By Distributive law

$\Leftrightarrow T_o \wedge (p \vee q \vee r)$ — By Inverse and Associative law

$\Leftrightarrow (p \vee q \vee r)$ — By Commutative law

(ii) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

We have, $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow \neg (\neg p \vee \neg q) \vee (p \wedge q \wedge r)$

Because $(u \rightarrow v) \Leftrightarrow (\neg u \vee v)$

$\Leftrightarrow (p \wedge q) \vee [(p \wedge q) \wedge r]$ — By De Morgan's law and Associative law

$\Leftrightarrow p \wedge q$ — By Absorption law

(iii) we have, $p \rightarrow (q \rightarrow r) \Leftrightarrow \neg p \vee (\neg q \vee r)$ — Because $(u \rightarrow v) \Leftrightarrow (\neg u \vee v)$

$\Leftrightarrow (\neg p \vee \neg q) \vee r$ — Associative law

$\Leftrightarrow (p \wedge q) \vee r$ — De-Morgan's law

$\Leftrightarrow (p \wedge q) \rightarrow r$ — Because $(u \rightarrow v) \Leftrightarrow (\neg u \vee v)$

Duality:

Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, the dual of s denoted by s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$ respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

Example: Given the primitive statements p, q, r and the compound statements

s: $(p \wedge (\neg q)) \vee (r \wedge T_0)$

s^d: $(p \vee (\neg q)) \wedge (r \vee F_0)$

Principle of Duality:

Let s and t be two statements that contains no logical connections than $\wedge$ and $\vee$**.** If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

Problems:

1. Write duals of the following propositions.
(i). $p \rightarrow q$ **(ii).** $(p \rightarrow q) \rightarrow r$ **(iii).** $p \rightarrow (q \rightarrow r)$

Solution: we recall that $(u \rightarrow v) \Leftrightarrow (\neg u \vee v)$

Therefore, by the principle of duality we find that

(i) $(p \rightarrow q)^d \Leftrightarrow (\neg p \vee q)^d \Leftrightarrow \neg p \wedge q$

(ii) $[(p \rightarrow q) \rightarrow r]^d \Leftrightarrow [\neg(\neg p \vee q) \vee r]^d$

$\Leftrightarrow [(p \wedge \neg q) \vee r]^d$

$\Leftrightarrow (p \vee \neg q) \wedge r$

(iii) $[p \rightarrow (q \rightarrow r)]^d \Leftrightarrow [\neg p \vee (q \rightarrow r)]^d$

$\Leftrightarrow [\neg p \vee (\neg q \vee r)]^d$

$\Leftrightarrow \neg p \wedge (\neg q \wedge r)$

2. Write duals of the following propositions.
(i). $q \rightarrow p$ **(ii).** $(p \vee q) \wedge r$ **(iii).** $(p \wedge q) \vee T_0$
(iv). $p \rightarrow (q \wedge r)$ **(v).** $p \leftrightarrow q$ **(vi).** $p \veebar q$

Solution: we recall that $(u \rightarrow v) \Leftrightarrow (\neg u \vee v)$

Therefore, by the principle of duality we find that

(i) $(q \rightarrow p)^d \Leftrightarrow (\neg q \vee p)^d \Leftrightarrow \neg q \wedge p$

(ii) $[(p \vee q) \wedge r]^d \Leftrightarrow (p \wedge q) \vee r$

(iii) $[(p \wedge q) \vee T_0]^d \Leftrightarrow (p \vee q) \wedge F_0$

(iv) $[p \rightarrow (q \wedge r)]^d \Leftrightarrow [\neg p \vee (q \wedge r)]^d \Leftrightarrow \neg p \wedge (q \vee r)$

(v) $[p \leftrightarrow q]^d \Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]^d \Leftrightarrow [(\neg p \vee q) \wedge (\neg q \vee p)]^d$

$\Leftrightarrow (\neg p \wedge q) \vee (\neg q \wedge p)$

(vi) $[p \veebar q]^d \Leftrightarrow [(p \wedge \neg q) \vee (q \wedge \neg p)]^d \Leftrightarrow [(p \vee \neg q) \wedge (q \vee \neg p)]$

NAND and NOR:

The compound proposition $\neg (p \wedge q)$ is read as "Not p and q" and also denoted by $(p \uparrow q)$. The symbol $\uparrow$ is called NAND connective.

The compound proposition $\neg (p \vee q)$ is read as "Not p or q" and also denoted by $(p \downarrow q)$. The symbol $\downarrow$ is called the NOR connective.

Truth table

p	q	$p \uparrow q$	$p \downarrow q$
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

Where $p \uparrow q = \neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$ and $p \downarrow q = \neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Problems:

1. For any propositions p, q Prove the following
(i). $\neg (p \downarrow q) \Leftrightarrow \neg p \uparrow \neg q$ **(ii).** $\neg (p \uparrow q) \Leftrightarrow \neg p \downarrow \neg q$

Solution: Using definition, we find that

i. $\neg (p \downarrow q) \Leftrightarrow \neg [\neg (p \vee q)]$
$\Leftrightarrow \neg [\neg p \wedge \neg q]$
$\Leftrightarrow \neg p \uparrow \neg q$

ii. $\neg (p \uparrow q) \Leftrightarrow \neg [\neg (p \wedge q)]$
$\Leftrightarrow \neg [\neg p \vee \neg q]$

$\Leftrightarrow \neg p \downarrow \neg q$

2. For any propositions p, q, r Prove the following

(i). $p \uparrow (q \uparrow r) \Leftrightarrow \neg p \vee (q \wedge r)$

(ii). $(p \uparrow q) \uparrow r \Leftrightarrow (p \wedge q) \vee \neg r$

(iii). $p \downarrow (q \downarrow r) \Leftrightarrow \neg p \wedge (q \vee r)$

(iv). $(p \downarrow q) \downarrow r \Leftrightarrow (p \vee q) \wedge \neg r$

Solution: Using definition, we find that

(i). $p \uparrow (q \uparrow r) \Leftrightarrow \neg [p \wedge (q \uparrow r))]$

$\Leftrightarrow \neg [p \wedge \neg (q \wedge r)]$

$\Leftrightarrow \neg p \vee \neg [\neg (q \wedge r)]$

$\Leftrightarrow \neg p \vee (q \wedge r)$

(ii). $(p \uparrow q) \uparrow r \Leftrightarrow \neg [(p \uparrow q) \wedge r]$

$\Leftrightarrow \neg [\neg (p \wedge q) \wedge r]$

$\Leftrightarrow \neg [\neg (p \wedge q)] \vee \neg r$

$\Leftrightarrow (p \wedge q) \vee \neg r$

(iii). $p \downarrow (q \downarrow r) \Leftrightarrow \neg [p \vee (q \downarrow r))]$

$\Leftrightarrow \neg [p \vee \neg (q \vee r)]$

$\Leftrightarrow \neg p \wedge \neg [\neg (q \vee r)]$

$\Leftrightarrow \neg p \wedge (q \vee r)$

(iv). $(p \downarrow q) \downarrow r \Leftrightarrow \neg [(p \downarrow q) \vee r]$

$\Leftrightarrow \neg [\neg (p \vee q) \vee r]$

$\Leftrightarrow \neg [\neg (p \vee q)] \wedge \neg r$

$\Leftrightarrow (p \vee q) \wedge \neg r$

Converse, Inverse and Contrapositive:

Consider a conditional $p \rightarrow q$ then:

1. $q \rightarrow p$ is called the converse of $p \rightarrow q$.
2. $\neg p \rightarrow \neg q$ is called the inverse of $p \rightarrow q$.
3. $\neg q \rightarrow \neg p$ is called the contrapositive of $p \rightarrow q$.

Truth table for converse, inverse and contrapositive

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

Note: 1. A conditional and its contrapositive are logically equivalent i.e., $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

2. A converse and the inverse of a conditional are logically equivalent

$$q \rightarrow p \Leftrightarrow \neg p \rightarrow \neg q$$

Logical implication:

Logical implication is a type of relationship between two statements or sentences. The relation translates verbally into "logically implies" or "if/then" and is symbolized by a double-lined arrow pointing toward the right ($\Rightarrow$). If p and q represent statements, then $p \Rightarrow q$ means "p implies q" or "If p, then q." The word "implies" is used in the strongest possible sense.

Example:

Suppose the sentences p and q are assigned as follows:

p = The sky is overcast.
q = The sun is not visible.

In this instance, $p \Rightarrow q$ is a true statement (assuming we are at the surface of the earth, below the cloud layer.) However, the statement $p \Rightarrow q$ is not necessarily true; it might be a clear night. Logical implication does not work both ways. However, the sense of logical implication is reversed if both statements are negated. i.e., $(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$

Using the above sentences as examples, we can say that if the sun is visible, then the sky is not overcast. This is always true. In fact, the two statements $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$ are logically equivalent.

Necessary and Sufficient Conditions:

Consider two propositions p and q whose truth values are interrelated. Suppose that $p \Rightarrow q$. Then in order that q may be true it is sufficient that p is true. Also, if p is true then it is necessary that q is true. In view of this interpretation, all of the following statements are taken to carry the same meaning:

(i). $p \Rightarrow q$ (ii). p is sufficient for q (iii). q is necessary for p

Problems:

1. State the converse inverse and contrapositive of
i) If the triangle is not isosceles, then it is not equilateral
ii) If the real number x^2 is greater than zero, then x is not equal to zero.
iii) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

Solution:

(i) p: Triangle is not isosceles and q: Triangle is not equilateral.

Implication: $p \rightarrow q$. if triangle is not isosceles then it is not equilateral.
Converse: $q \rightarrow p$. if a triangle is not equilateral then it is not isosceles.
Inverse: $\neg p \rightarrow \neg q$. if a triangle is isosceles then it is equilateral.
Contrapositive: $\neg q \rightarrow \neg p$: if a triangle is equilateral then it is isosceles.

(ii) p: A real number x^2 is greater than zero and q: x is not equal to zero.
Implication: $p \rightarrow q$. if a real number x^2 is greater than zero then, x is not equal to zero.
Converse: $q \rightarrow p$. if a real number x is not equal to zero then, x^2 is greater zero.
Inverse: $\neg p \rightarrow \neg q$. if a real number x^2 is not greater than zero then, x is equal to zero.
Contrapositive: If a real number x is equal to zero then, x^2 is not greater than zero

(iii) p: If Quadrilateral is a parallelogram and q: its Diagonals Bisects each other.

Implication: $p \rightarrow q$. If Quadrilateral is a parallelogram, then its diagonals bisects each other.
Converse: $q \rightarrow p$. If the diagonals of the Quadrilateral bisect each other, then it is a parallelogram.
Inverse: $\neg p \rightarrow \neg q$. If Quadrilateral is not a parallelogram, then its diagonals do not bisect each other.
Contrapositive: $\neg q \rightarrow \neg p$: If the diagonals of the Quadrilateral do not bisect each other, then it is a not a parallelogram.

2. Write down the following statements in the 'Necessary and Sufficient Condition' Language.
i) If the triangle is not isosceles, then it is not equilateral
ii) If the real number x^2 is greater than zero, then x is not equal to zero.
iii) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

Solution:

Necessary Condition Language:

(i). For a triangle to be non-isosceles it is necessary that is not equilateral.

(ii). A necessary condition for a real number x^2 to be greater than zero is that x is not equal to zero.

(iii). A necessary condition for a quadrilateral to be a parallelogram is that its diagonals bisect each other.

Necessary Condition Language:

(i). A sufficient condition for a triangle to be not equilateral is that it is not isosceles.

(ii). For a real number x, the condition x^2 to be greater than zero is sufficient for x to be not equal to zero.

(iii). A sufficient condition for the diagonals of a quadrilateral to bisect each other is that the quadrilateral is a parallelogram.

☻ **<u>Rules of inference</u>:**

Let us consider the implication $(p_1 \wedge p_2 \wedge \ldots . \wedge p_n) \rightarrow q$

Here n is a positive integer, the statements $p_1, p_2, \ldots . p_n$ are called the **premises of the argument** and q is called the **conclusion of the argument**.

We write the above argument in the following tabular form:

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \vdots \\ \underline{p_n} \\ \therefore q \end{array}$$

The preceding argument is said to be valid if whenever each of the premises $p_1, p_2, \ldots . p_n$ is true, then the conclusion q is likewise true.
i.e., $(p_1 \wedge p_2 \wedge \ldots . \wedge p_n) \rightarrow q$ is valid when $(p_1 \wedge p_2 \wedge \ldots . \wedge p_n) \Rightarrow q$
It is to be emphasized that in an argument, the premises are always taken to be true whereas the conclusion may be true or false. The conclusion is true only in the case of valid argument. There exist rules of logic which can be employed for establishing the validity of arguments. These rules are called Rules of Inference.

Name of the rule and rule of inference

Sl.no	Rules of inference	Name of rule
1	$\begin{array}{c} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$	Rule of Detachment (modus pones)
2	$\begin{array}{c} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$	Law of Syllogism
3	$\begin{array}{c} p \rightarrow q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$	Modus Tollens
4	$\begin{array}{c} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$	Rule of Conjunction
5	$\begin{array}{c} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$	Rule of Disjunctive Syllogism
6	$\begin{array}{c} \underline{\neg p \rightarrow F_0} \\ \therefore p \end{array}$	Rule of Contradiction
7	$\begin{array}{c} \underline{p \wedge q} \\ \therefore p \end{array}$	Rule of Conjunctive Simplification
8	$\begin{array}{c} \underline{p} \\ \therefore p \vee q \end{array}$	Rule of Disjunctive Amplification

Problems:

1. Test whether the following is valid argument.

If Sachin hits a century, then he gets a free car.

Sachin hits a century.

∴ Sachin gets a free car.

Solution: Let p: Sachin hits a century.

q: Sachin gets a free car.

The given statement reads

$$\begin{array}{r} p \rightarrow q \\ \underline{\quad p} \\ \therefore q \end{array}$$

In view of Modus Pones Rule, this is a valid argument.

2. Test the validity of the following arguments.
If Ravi goes out with friends, he will not study.
If Ravi does not study, his father will become angry.
His father is not angry.
∴ Ravi has not gone out with friends.

Solution: Let p: Ravi goes out with friends.

q: Ravi does not study.

r: His father gets angry.

Then the given argument reads.

$$\begin{array}{r} p \rightarrow q \\ q \rightarrow r \\ \underline{\quad \neg r} \\ \therefore \neg p \end{array}$$

This argument is logically equivalent to (Using the rule of syllogism)

$$\begin{array}{r} p \rightarrow r \\ \underline{\quad \neg r} \\ \therefore \neg p \end{array}$$

In view of Modus Tollens Rule, this is a valid argument.

3. Test whether the following is valid argument.

If Sachin hits a century, then he gets a free car.

Sachin does not get a free car.

∴ Sachin has not hit a century

Solution: Let p: Sachin hits a century.

q: Sachin gets a free car.

The given statement reads

$$\begin{array}{r} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

In view of Modus Tollens Rule, this is a valid argument.

4. Test the validity of the following argument
If I study, then I'll not fail in the examination.
If I do not watch tv in the evenings, I will study.
I failed in the examination.
∴ I must have watched tv in the evenings.

Solution: Let p: I study

q: I fail in the examination

r: I watch tv in the evenings.

Then the given argument reads

$$\begin{array}{r} p \rightarrow \neg q \\ \neg r \rightarrow p \\ q \\ \hline \therefore r \end{array}$$

This argument is logically equivalent to

$$\begin{array}{r} q \rightarrow \neg p \\ \neg p \rightarrow r \\ q \\ \hline \therefore r \end{array}$$

(because (p →¬ q) ⇔ (¬ ¬ q → ¬ p))

(because (¬ r → p) ⇔ (¬ p → r))

This is equivalent to (Using rule of syllogism)

$$\begin{array}{r} q \rightarrow r \\ q \\ \hline \therefore r \end{array}$$

In view of Modus Pones Rule, this is a valid argument.

5. Test the validity of the following argument
I will become famous or I will not become a musician.
I will become a musician.
∴ I will become famous.

Solution: Let p: I will become famous

q: I will become a musician

Then the given argument reads

$$\begin{array}{r} p \vee \neg q \\ q \\ \hline \therefore p \end{array}$$

This argument is logically equivalent to

$$\begin{array}{r} q \to p \\ q \\ \hline \therefore p \end{array}$$

Because $p \vee \neg q \Leftrightarrow \neg q \vee p \Leftrightarrow q \to p$

In view of Modus Pones Rule, this is a valid argument.

6. Test the validity of the following argument
I will get grade A in this course or I will not graduate.
If I do not graduate, I will join army.
I got grade A.
∴ I will not join army.

Solution: Let p: I will get grad A in this course

q: I do not graduate.

r: I will join army.

Then the given argument reads

$$\begin{array}{r} p \vee q \\ q \to r \\ p \\ \hline \therefore \neg r \end{array}$$

This argument is logically equivalent to

$$\begin{array}{r} \neg q \to p \\ \neg r \to \neg q \\ p \\ \hline \therefore \neg r \end{array}$$

Because $p \vee \neg q \Leftrightarrow q \vee p \Leftrightarrow \neg q \to p$ and using Contrapositive.

This is equivalent to (Using rule of syllogism)

$$\begin{array}{r} \neg r \to p \\ p \\ \hline \therefore \neg r \end{array}$$

This is not a valid argument.

7. Test whether the following is valid argument.

If Sachin hits a century, then he gets a free car.

<u>Sachin gets a free car.</u>

∴ Sachin has hit a century.

Solution: Let p: Sachin hits a century.

q: Sachin gets a free car.

The given statement reads

$$\begin{array}{r} p \to q \\ \underline{\quad q} \\ \therefore p \end{array}$$

We note that if $p \to q$ and q are true, there is no rule which asserts that p must be true. Indeed, p can be false when $p \to q$ and q are true. See the table below.

p	q	$p \to q$	$(p \to q) \wedge q$
0	1	1	1

Thus, $[(p \to q) \wedge q] \to p$ is not a tautology. Hence, this is not a valid argument.

8. Test the Validity of the following argument:

(i). $p \wedge q$

$\underline{p \to (q \to r)}$

$\therefore\ r$

(ii). p

$p \to \neg q$

$\underline{\neg q \to \neg r}$

$\therefore\ \neg r$

(iii). $p \to r$

$\underline{q \to r}$

$\therefore\ (p \vee q) \to r$

Solution:

(i). Since $p \wedge q$ is true, both p and q are true. Since p is true and $p \to (q \to r)$ is true, $q \to r$ Should be true. Since q is true and $q \to r$ is true, r should be true. Hence the given argument is valid.

(ii). The premises $p \to \neg q$ and $\neg q \to \neg r$ together yields the premise $p \to \neg r$. since p is true, this premise $p \to \neg r$ establishes that $\neg r$ is true. Hence the given argument is valid.

(iii) We note that

$(p \to r) \wedge (q \to r) \Leftrightarrow (\neg p \vee r) \wedge (\neg q \vee r)$

$\Leftrightarrow (r \vee \neg p) \wedge (r \vee \neg q)$ By Commutative law

$\Leftrightarrow r \vee (\neg p \wedge \neg q)$ By Distributive law

$\Leftrightarrow \neg (p \vee q) \vee r$ By Commutative & De Morgan's Law

$\Leftrightarrow (p \vee q) \to r$

This Logical equivalence shows that the given argument is valid.

9. Test whether the following arguments are valid:

(i).

$$\begin{array}{l} p \to q \\ r \to s \\ \underline{p \lor r} \\ \therefore\ q \lor s \end{array}$$

(ii).

$$\begin{array}{l} p \to q \\ r \to s \\ \underline{\neg q \lor \neg s} \\ \therefore \neg (p \land r) \end{array}$$

Solution:

(i) We note that

$$\begin{aligned}
(p \to q) \land (r \to s) \land (p \lor r) &\Leftrightarrow (p \to q) \land (r \to s) \land (\neg p \to r) \\
&\Leftrightarrow (p \to q) \land (\neg p \to r) \land (r \to s) && \text{By Commutative law} \\
&\Leftrightarrow (p \to q) \land (\neg p \to s) && \text{Using Rule of Syllogism} \\
&\Leftrightarrow (\neg q \to \neg p) \land (\neg p \to s) && \text{Using Contrapositive} \\
&\Leftrightarrow (\neg q \to s) && \text{Using Rule of Syllogism} \\
&\Leftrightarrow q \lor s
\end{aligned}$$

This Logical equivalence shows that the given argument is valid.

(ii) We note that

$$\begin{aligned}
(p \to q) \land (r \to s) \land (\neg q \lor \neg s) &\Leftrightarrow (p \to q) \land (r \to s) \land (q \to \neg s) \\
&\Leftrightarrow (p \to q) \land (q \to \neg s) \land (r \to s) && \text{By Commutative law} \\
&\Leftrightarrow (p \to \neg s) \land (r \to s) && \text{Using Rule of Syllogism} \\
&\Leftrightarrow (p \to \neg s) \land (\neg s \to \neg r) && \text{Using Contrapositive} \\
&\Leftrightarrow (p \to \neg r) && \text{Using Rule of Syllogism} \\
&\Leftrightarrow \neg p \lor \neg r) \\
&\Leftrightarrow \neg (p \land r)
\end{aligned}$$

This Logical equivalence shows that the given argument is valid.

10. Show that the following argument is not valid:

$$\begin{array}{l} p \\ p \lor q \\ q \to (r \to s) \\ \underline{t \to r \qquad\qquad} \\ \therefore \neg s \to \neg t \end{array}$$

Solution:

Here p is true (premise) and $(p \lor q)$ is true (premise). Therefore, q may be true or false.

Suppose q is false. Then, since $q \to (r \to s)$ is true (premise), $r \to s$ must be false. This means that r must be true, and s must be false. Since r is true and $t \to r$ is true (premise), t may be true or false. Suppose t is true, then $\neg t$ is false. Since s must be false, $\neg s$ must be true. Consequently, $\neg s \to \neg t$ is false.

Thus, when q is false and t is true, the given conclusion does not follow from the given premise. As such, the given argument is not valid argument.

☻ **Open statement:**

A declaration statement is an open statement

i. If it contains one or more variables.
ii. If it is not statement.
iii. But it becomes statement when the variables in it are replaced by certain allowable choices.

Example: "The number x+2 is an even integer" is denoted by P(x) then ¬ P(x) may be read as "The number x+2 is not an even integer".

Quantifiers:

The words "all", "every", "some", "there exist" are associated with the idea of a quantity such words are called quantifiers.

1. **Universal quantifiers**:
   The symbol ∀ has been used to denote the phrases "for all" and "for every" in logic "for each" and "for any" are also taken up to equivalent to these. These equivalent phrases are called universal quantifiers.
2. **Existential quantifiers**:
   The symbol ∃ has been used to denote the phrases "there exist", "for some" and "for at least one" each of these equivalent phrases is called the existential quantifiers.

Example: 1. For every integer x, x^2 is a non-negative integer $\exists\, x \in s, P(x)$.

2. For the universe of all integers, let
   p(x): x>0.
   q(x): x is even.
   r(x): x is a perfect square.
   s(x): x is divisible by 3.
   t(x): x is divisible by 7.

Problems:

Write down the following quantified statements in symbolic form:

i) At least one integer is even.

ii) There exists a positive integer that is even.

iii) Some integers are divisible by 3.

iv) every integer is either odd or even.

v) if x is even and a perfect square, then is not divisible by 3.

vi) if x is odd or is not divisible by 7, then x is divisible by 3.

Solution:

Using the definition of quantifiers, we find that the given statement read as follows in symbolic form

i) $\exists x, q(x)$

ii) $\exists x, [\ p(x) \wedge q(x)]$

iii) $\exists x, [\ q(x) \wedge s(x)]$

vi) $\forall x, [\ q(x) \veebar \neg q(x)]$

v) $\forall x\ [\ \{q(x) \wedge r(x)\} \rightarrow s(x)]$

vi) $\forall x, [\{\neg q(x) \vee \neg t(x)\} \rightarrow s(x)]$

<u>Rules employed for determining truth value</u>:

Rule1: The statement "$\forall\ x \in s, p(x)$" is true only when $p(x)$ is true for each $x \in s$.

Rule2: The statement "$\exists\ x \in s, p(x)$" is false only when $p(x)$ is false for every $x \in s$.

*****<u>Rules of inference</u>:

Rule3: If an open statement $p(x)$ is known to be true for all x in a universe s and if $a \in s$ then $p(a)$ is true. (this is known as the rule of universal specification).

Rule4: if an open statement $p(x)$ is proved to be true for any (arbitrary) x chosen from a set s then the quantified statement $\forall\ x \in s, p(x)$ is true. (this is known as the rule of universal generalization)

*****<u>Logical equivalence</u>:

Two quantified statements are said to be logically equivalent whenever they have the same truth values in all possible situations.

The following results are easy to prove.

i) $\forall\ x\ [p(x) \wedge q(x)] \Leftrightarrow (\forall\ x, p(x)) \wedge (\forall\ x, q(x))$

ii) $\exists\ x\ [p(x) \vee q(x)] \Leftrightarrow (\exists\ x, p(x)) \vee (\exists\ x, q(x))$

iii) $\exists\ x, [p(x) \rightarrow q(x)] \Leftrightarrow \exists\ x, (\neg p(x) \vee q(x))$

*****<u>Rule for negation of a quantified statement</u>:

Rule5: To construct the negation of a quantified statement, change the quantifier from universal

to existential and vice versa.

i.e., $\neg [\forall\ x, p(x)] \equiv \exists\ x, [\ \neg p(x)]$

$\neg [(\exists\ x, p(x)] \equiv \forall\ x\ [\ \neg p(x)]$

Problems:

1. Consider the open statements with the set of real numbers as the universe.

$p(x): |x|>3$, $q(x): x>3$

Find the truth value of the statement $\forall x, [p(x) \rightarrow q(x)]$. Also, write down the converse, inverse and the contrapositive of this statement and find their truth values

Solution:

We readily note that

$p(-4) \equiv |-4|>3$ is true and $q(-4) \equiv -4>3$ is false

Thus, $p(x) \rightarrow q(x)$ is false for $x= -4$.

Accordingly, the given statement $\forall x, [p(x) \rightarrow q(x)]$ .... (i) is false.

The converse of the statement (i) is $\forall x, [q(x) \rightarrow p(x)]$ ....(ii)

In words, this reads "For every real number x, x>3 then |x|>3"

Or Equivalently, "Every real number greater than 3 has its absolute value (magnitude) greater than 3"

This is a true statement.

Next, the inverse of the statement (i) is $\forall x, [\neg p(x) \rightarrow \neg q(x)]$ ....(iii)

In words this reads "For every real number x, if $|x| \leq 3$ then $x \leq 3$"

Or equivalently, "If the magnitude of a real number is less than or equal to 3, then the number is less than or equal to 3"

Since the converse and inverse of a conditional are logically equivalent the statements (ii) and (iii) have the same truth values. Thus iii) is a true statement.

Then the contrapositive of statement (i) is $\forall x, [\neg q(x) \rightarrow \neg p(x)]$...........................(iv)

"Every real number which is less than or equal to 3 has its magnitude less than or equal to 3".

2. Let $p(x): x^2-7x+10$, $q(x): x^2-2x-3$, $r(x): x<0$.
Determine the truth or falsity of the following statements. When the universe U contains only the integers 2 and 5. If a statement is false. Provide a counter example or explanation.

(i). $\forall x, [p(x) \rightarrow \neg r(x)]$ **(ii).** $\forall x, [q(x) \rightarrow r(x)]$
(iii). $\exists x, [q(x) \rightarrow r(x)]$ **(iv).** $\exists x, [p(x) \rightarrow r(x)]$

Solution:

Here, the universe is $U= \{2, 5\}$.

We note that $x^2-7x+10 = (x-5)(x-2)$. Therefore, p(x) is true for x=5 and 2. That is p(x) is true for all $x \in U$.

Further, x^2-2x-3 = (x-3) (x+1). Therefore, q(x) is only true for x=3 and x=-1. Since x=3 and x=-1 are not in the universe, q(x) is false for all x ∈ U

Obviously, r(x) is false for all x ∈ U.

Accordingly:

(i) Since p(x) is true for all x ∈ U and ¬ r(x) is true for all x ∈ U, the statement ∀ x, [p(x) → ¬ r(x)] is true.

(ii) Since q(x) is false for all x ∈ U and r(x) is false for all x ∈ U, the statement ∀ x, [q(x) → r(x)] is true.

(iii) Since q(x) and r(x) are false for x=2, the statement ∃ x, [q(x) → r(x)] is true.

(iv) Since p(x) is true for all x ∈ U but r(x) is false for all x ∈ U. the statement p(x) → r(x) is false for all x ∈ U. consequently, ∃ x, [p(x) → r(x)] is false.

3. Negate and simplify each of the following.

(i). ∃ x, [p(x) ∨ q(x)] **(ii).** ∀ x, [p(x) ∧ ¬ q(x)]
(iii). ∀ x, [p(x) → q(x)] **(iv).** ∃ x, [p(x) ∨ q(x)] → r(x)

Solution:

By using the rule of negation for quantified statements and the laws of logic, we find that

(i) ¬ [∃ x, {p(x) ∨ q(x)}] ≡ ∀ x, [¬ {p(x) ∨ q(x)]

≡ ∀ x, [¬ p(x) ∧ ¬ q(x)]

(ii) ¬ [∀ x, {p(x) ∧ ¬ q(x)}] ≡ ∃ x, [¬ {p(x) ∧ ¬ q(x)}]

≡ ∃ x, [¬ p(x) ∨ q(x)]

(iii) ¬ [∀ x, {p(x) → q(x)}] ≡ ∃ x, [¬ {¬ p(x) ∨ q(x)}]

≡ ∃ x, [p(x) ∧ ¬ q(x)}]

(iv) ¬ [∃ x, {p(x) ∨ q(x)} →r(x)] ≡ ∀ x, [¬ {¬ (p(x) ∨ q(x)) ∨ r(x)}]

≡ ∀ x, [{p(x) ∨ q(x)} ∧ ¬ r(x)]

4. Write down the following proposition in symbolic form, and find its negation:
"If all triangles are right angled, then no triangle is equiangular".

Solution:

Let T denote set of all triangles. Also, p(x): x is right angled, q(x): x is equiangular.

Then in symbolic form, the given proposition reads

{∀ x ∈ T, p(x)} → {∀ x ∈ T, ¬ q(x)}

The negation of this is

$\{\forall\, x \in T, p(x)\} \wedge \{\exists\, x \in T, q(x)\}$

In words, this reads "All triangles are right angled and some triangles are equiangular".

Logical implication involving quantifiers

5. Prove that $\exists\, x, [p(x) \wedge q(x)] \Rightarrow \exists\, x, p(x) \wedge \exists\, x, q(x)$

Is the converse true

Solution:

Let S denote the universe, we find that

$\exists\, x, [p(x) \wedge q(x)] \Rightarrow p(a) \wedge q(a)$ for some $a \in S$

$\Rightarrow p(a)$, for $a \in S$ and $q(a)$ for some $a \in S$

$\Rightarrow \exists\, x, p(x) \wedge \exists\, x, q(x)$

This proves the required implication.

Next, we observe that $\exists\, x, p(x) \Rightarrow p(a)$ for some $a \in S$ and $\exists\, x, q(x) \Rightarrow q(b)$ for some $b \leftarrow S$.

Therefore, $\exists\, x, p(x) \wedge \exists\, x, q(x) \Rightarrow p(a) \wedge q(b)$

$\not\Leftrightarrow p(a) \wedge q(a)$ because b need not be a

Thus, $\exists\, x, [p(x) \wedge q(x)]$ need not be true when $\exists\, x, p(x) \wedge \exists\, x, q(x)$ is true.

That is $\exists\, x, p(x) \wedge \exists\, x, q(x) \not\Leftrightarrow [p(x) \wedge q(x)]$

Accordingly, the converse of the given implication is not necessarily true.

6. Find whether the following arguments is valid:

No engineering student of first or second semester studies logic

<u>Anil is a student who studies logic</u>.

∴ Anil is not in second semester

Solution:

Let us take the universe to be the set of all engineering students

p(x): x is in first semester.

q(x): x is in second semester.

r(x): x studies logic.

Then the given argument reads

$$\begin{array}{c} \forall\, x,[\{p(x) \vee q(x)\} \rightarrow \neg\, r(x)] \\ r(a) \\ \hline \therefore \neg\, q(a) \end{array}$$

We note that

$\forall x, [\{p(x) \lor q(x)\} \rightarrow \neg r(x)] \Rightarrow \{p(a) \lor q(a)\} \rightarrow \neg r(a)$

By rule of universal specification.

Therefore,

$[\forall x, \{p(x) \lor q(x)\} \rightarrow \neg r(x)] \wedge r(a)$

$\Rightarrow [\{p(a) \lor q(a)\} \rightarrow \neg r(a)] \wedge r(a)$

$\Rightarrow r(a) \wedge [r(a) \rightarrow \neg [p(a) \lor q(a)]]$, Using Commutative law and Contrapositive

$\Rightarrow \neg [p(a) \lor q(a)]$, By the Modus Pones law

$\Rightarrow \neg p(a) \wedge \neg q(a)$, By De Morgan's law

$\Rightarrow \neg q(a)$, by the rule of conjunctive specification,

This proves that the given argument is valid.

7. Find whether the following argument is valid.
If a triangle has 2 equal sides then, it is isosceles.
If the triangle is isosceles, then it has 2 equal angles.
<u>A certain triangle ABC does not have 2 equal angles.</u>
∴ the triangle ABC does not have 2 equal sides.

Solution:

Let the universe be set of all triangles

And let $p(x)$:x has equal sides.

$q(x)$: x is isosceles.

$r(x)$: x has 2 equal angles.

Also let C denote the triangle ABC.

Then, in symbols, the given argument reads as follows:

$\forall x, [p(x) \rightarrow q(x)]$

$\forall x, [q(x) \rightarrow r(x)]$

$\underline{\neg r(c)}$

$\therefore p(c)$

We note that

$\forall x, [p(x) \rightarrow q(x)] \wedge \{\forall x, [q(x) \rightarrow r(x)]\} \wedge \neg r(c)$

$\Rightarrow \{\forall x, [p(x) \rightarrow r(x)] \wedge \neg r(c)\}$, By Rule of Syllogism

$\Rightarrow \{[p(c) \rightarrow r(c)] \wedge \neg r(c)\}$, By Rule of Universal Specification

$\Rightarrow \neg p(c)$ By Modus Tollens Rule

This proves that the given argument is valid.

8. Prove that the following argument is valid.

$\forall x, [p(x) \lor q(x)]$
$\exists x, \neg p(x)$
$\forall x, [\neg q(x) \lor r(x)]$
$\forall x, [s(x) \rightarrow \neg r(x)]$
$\therefore \exists x, \neg s(x)$

Solution:

We note that

$\{\forall x, [p(x) \lor q(x)]\} \wedge [\exists x, \neg p(x)]$
$\Rightarrow [p(a) \lor q(a)] \wedge \neg p(a)$ For some as in the universe
$\Rightarrow q(a)$ By Disjunctive Syllogism

Therefore, $\{\forall x, [p(x) \lor q(x)]\} \wedge [\exists x, \neg p(x)] \wedge \{\forall x, [\neg q(x) \lor r(x)]\}$
$\Rightarrow q(a) \wedge [\neg q(a) \lor r(a)]$
$\Rightarrow r(a)$ By Rule of Disjunctive Syllogism

Consequently,

$\{\forall x, [p(x) \lor q(x)]\} \wedge \{\exists x, \neg p(x)\} \wedge \{\forall x, [\neg q(x) \lor r(x)]\} \wedge \{\forall x, [s(x) \rightarrow \neg r(x)]\}$
$\Rightarrow r(a) \wedge \{s(a) \rightarrow \neg r(a)\}$
$\Rightarrow \neg s(a)$ By Modus Tollens rule
$\Rightarrow \exists x, \neg s(x)$.

This proves the given argument is valid.

Quantified statements with more than one variable

9. Determine the truth value of each of the following quantified statements. The universe being the set of all non-zero integer.

i) $\exists x, \exists y\, [xy=1]$
ii) $\exists x\, \forall y\, [xy=1]$
iii) $\forall x\, \exists y\, [xy=1]$
iv) $\exists x, \exists y, [(2x+y=5) \wedge (x-3y=-8)]$
v) $\exists x, \exists y, [(3x-y=17) \wedge (2x+4y=3)]$

Solution: (i) true (take x=1, y=1)

(ii) False (for specified x, xy=1 for every y is not true)

(iii) false (for x=2, there is no integer y such that xy=1)

(iv) true (take x=1, y=3)

(v) false (equation 3x-y=17 and 2x+4y=3 do not have a common integer solution)

☻ **Methods of proof and methods of disproof:**

Direct proof:

1. **Hypothesis:** first assume that p is true.
2. **Analysis:** starting with the hypothesis and employ the rules/ Laws of logic and other known facts infer that q is true.
3. **Conclusion:** $p \rightarrow q$ is true.

Indirect proof:

A conditional $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ is logically equivalent. In some situations, given a condition $p \rightarrow q$, a direct proof of the contrapositive $\neg q \rightarrow \neg p$ is easier. On the basis of this proof, we infer that the conditional $p \rightarrow q$ is true. This method of proving a conditional is called an indirect method of proof.

Proof by contradiction:

1. **Hypothesis:** assume that $p \rightarrow q$ is false, that is assume that p is true and q is false.
2. **Analysis:** starting with the hypothesis that q is false and employing the rules of logics and other known facts, this infer that p is false. This contradicts the assumption that p is true.
3. **Conclusion:** because of the contradiction arrived in the analysis, we infer that $p \rightarrow q$ is true.

Proof by exhaustion:

Recall that a proposition of the form "$\forall x \in S, p(x)$" is true if p(x) is true for every x in S. if S consists of only a limited number of elements, we can prove that the statement "$\forall x \in S, p(x)$" is true by considering p(a) for each a in S and verifying that p(a) is true (in each case). Such a method of proof is called the method of exhaustion.

Disproof by counter example:

The way of disproving a proposition involving the universal quantifiers is to exhibit just one case where the proposition is false. This method of disproof is called disproof by counter example.

Problems:

1. Prove that, for all integers k and l, if k and l are both odd the k+l is even and kl is odd.

Solution:

Take any two integers k and l, and assume that both of these are odd (hypothesis)

Then k=2m+1, l=2n+! for some integers m and n. therefore,

$$k+1= (2m+1) +(2n+1) = 2(m+n+1)$$

$$kl= (2m+1)(2n+1) = 4mn+2(m+n)+1$$

We observe that k+l is divisible by 2 and kl is not divisible by 2. Therefore k+l is an even integer and kl is an odd integer.

Since k and l are arbitrary integers, the proof of the given statement is complete.

2. For each of the following statements, provide an indirect proof by stating and proving the contrapositive of the given statement.
(i) for all integers k and l, if kl is odd then both k and l are odd.
(ii) for all integers k and l if k+l is even, then k and l are both even or both odd.

Solution:

The contrapositive of the given statement is

"For all integers k and l, if k is even or l is even then kl is even.

We now prove this contrapositive.

For any integers k and l, assume that k is even.

Then k=2m for some integer m, and kl=(2m)l=2(ml) which is evidently even. Similarly if l is even, then kl=k(2n)=2kn for some integer n so that kl is even. This proves the contrapositive.

This proof of contrapositive serves as an indirect proof of the given statement.

(ii). Here, the contrapositive of the given statement is

"for all integers k and l, if one of k and l and is odd and the other is even, then k+l is odd"

We now prove this contrapositive

For any odd integers k and l, assume that, one of k and l is odd and the other is even.

Suppose k is odd and l is even. Then k=2m+1 and l=2n for some integers m and n. consequently k+l=(2m+1)+2n which is evidently odd.

Similarly, if k is even and l is odd, we find that k+l is odd. This proves the contrapositive.

This proof of contrapositive serves as an indirect proof of the given statement.

3. Give (i) direct proof (ii) indirect proof (iii) proof by contradiction for the following statement: "if n is an odd integer, then n+9 is an even integer".

Solution:

(i) Direct proof: assume that n is an odd integer. Then n=2k+1 for some integer k. This gives n+9 = (2k+1)+9 = 2(k+5) from which it is evident that n+9 is even. This establishes the truth of the given statement by a direct proof.

(ii) Indirect proof: assume that n+9 is not an even integer. Then n+9 = 2k+1 for some integer k. This gives n = (2k+1)-9=2(k-4), which shows that n is even. Thus, if n+9 is not even, then n is not odd. This proves the contrapositive of the given statement. This proof of the contrapositive serves as an indirect proof of the given statement.

(iii) proof by contradiction: assume that the given statement is false. That is, assume that n is odd and n+9 is odd, n+9=2k+1 for some integer k so that n=(2k+1)-9= 2(k-4) which shows that n is even. This contradicts the assumption that n is odd. Hence the given statement must be true.

4. Prove that every even integer n with $2 \leq n \leq 26$ can be written as a sum of most three perfect squares.

Solution:

Let S={2, 4, 6, …., 24, 26}. We have to prove that the statement: "$\forall$ x $\in$ S, p(x)" is true, where p(x): x is a sum of at most three perfect squares.

We observe that

$2=1^2+1^2$	$16=4^2$
$4=2^2$	$18=4^2+1^2+1^2$
$6=2^2+1^2+1^2$	$20=3^2+3^2+1^2+1^2$
$8=2^2+2^2$	$22=3^2+3^2+2^2$
$10=3^2+1^2$	$24=4^2+2^2+2^2$
$12=2^2+2^2+2^2$	$26=5^2+1^2$
$14=3^2+2^2+1^2$	

The above facts verify that each x in S is a sum of at most three-perfect square.

5. Prove or disprove that the sum of square of any four non-zero integers is an even integer.

Solution:

Here the proposition is

"For any four non-zero integers a, b, c, d and $a^2+b^2+c^2+d^2$ is an even integer".

We check that for a=1, b=1, c=1, d=2 the proposition is false. Thus, the given proposition is not a true proposition. This proposition is disproved through the counter example a=b=c=1 and d=2.

6. Consider the following statement for the universe of integers if n is an integer then $n^2=n$ or $\forall n \{n^2=n\}$.

Solution:

For n=0 it is true that $n^2=0^2=0=n$ and if n=1 is also true that $n^2=1^2=1=n$. however we cannot conclude that $n^2=n$ for every integer n.

The rule of universal generalisation does not apply here, for we cannot consider the choices of 0 (or 1) as an arbitrarily chosen integer. If n=2, $n^2=4 \neq n=2$, and this one counter example is enough to tell us that the given statement is false.

However, either replacement namely n=0 or n=1 is not enough to establish the truth of the statement. For some integer n, $n^2=n$ or $\exists n \{n^2=n\}$.

7. For all positive integers x and y if the product xy exceeds 25, then x>5 or y>5.

Solution:

Consider the negation of the conclusion that is suppose that $0 < x \leq 5$ and $0 < y \leq 5$. Under these circumstances we find that $0 < x \cdot y < 5 \cdot 5 = 25$.

So, the product of xy does not exceed 25.